# THE FRESNEL DIFFRACTION INTEGRAL IS A FRACTIONAL FOURIER TRANSFORM

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The propagation of light can be viewed as a process of continual fractional Fourier transformation. As light propagates, its amplitude distribution evolves through fractional transforms of increasing order.

Namias introduced the fractional-order Fourier transform [1] and McBride and Kerr provided mathematical rigor and the definition for the operator for all orders, $\alpha \in \mathbb{R}$ [2]. Pellat-Finet related the two-dimensional fractional Fourier transform to Fresnel diffraction, connecting the composition of the operator with the Huygens principle [3]. Ozaktas and Mendlovic explored the relationshp between wave amplitudes on spherical surfaces of certain radii and separation, as well as the description of general optical systems consisting of sequences of lenses and free-space segments [4]. More recently, Schnebelin and Guillet de Chatellus used the fractional Fourier transform to describe the integer and fractional Talbot effects and their application to analog signal processing [5].

Almeida interpreted the order $\alpha$ as the angle of rotation in the time–frequency plane, and made connections to the Wigner distribution and the ambiguity function [6].

## 1. Derivation of the Huygens–Fresnel principle

This derivation follows §3.4–3.7 and §4.1.2–4.5 of Goodman [7]. Let $S_1$ be the planar surface just behind the diffracting aperture and let $S_2$ be the spherical surface of radius $R$ centered at $P_0$ that joins and closes $S = S_1 + S_2$. The vector $\mathbf{r}_{01}$ points from $P_0$ to $P_1$, where the latter is on $S_1$.

$$U(P_0) = \frac{1}{4\pi} \iint_{S_1+S_2} \left( G \frac{\partial U}{\partial n} - U \frac{\partial G}{\partial n} \right) \mathrm{d}s \tag{1}$$

where $G = \exp(jkr_{01})/r_{01}$. On $S_2$, $G = \exp(jkR)/R$. Also, given that

$$\frac{\partial G(P_1)}{\partial n} = \cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01}) \left( jk - \frac{1}{r_{01}} \right) \frac{\exp(jkr_{01})}{r_{01}} = \cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01}) \left( jk - \frac{1}{r_{01}} \right) G(P_1), \tag{2}$$

then

$$\frac{\partial G}{\partial n} = \left( jk - \frac{1}{R} \right) G \approx jkG, \tag{3}$$

where the approximation is valid for large $R$. The $S_2$ portion of the integral can be written

$$\iint_{S_2} \left( G \frac{\partial U}{\partial n} - U(jkG) \right) \mathrm{d}s = \int_{\Omega} G \left( \frac{\partial U}{\partial n} - jkU \right) R^2 \,\mathrm{d}\omega, \tag{4}$$

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where $\Omega$ is the solid angle subtended by $S_2$ at $P_0$. The magnitude of $RG$, $|RG| = |\exp(jkR)|$, is uniformly bounded on $S_2$. Thus this integral over $S_2$ will vanish as $R$ becomes large as long as

$$\lim_{R\to\infty} R\left(\frac{\partial U}{\partial n} - jkU\right) = 0, \tag{5}$$

which is the Sommerfeld radiation condition. This is satisfied if $U$ vanishes as fast as a diverging spherical wave, which is the case if there are only outgoing waves impinging on $S_2$. Thus,

$$\frac{1}{4\pi}\iint_{S_2}\left(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n}\right)\mathrm{d}s = 0 \tag{6}$$

and therefore

$$U(P_0) = \frac{1}{4\pi}\iint_{S_1}\left(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n}\right)\mathrm{d}s. \tag{7}$$

We assume the diffracting screen is opaque except for the aperture, $\Sigma$.

Now apply the Kirchhoff boundary conditions:[1]

(1) Across the surface $\Sigma$, the field distribution $U$ and its derivative $\partial U/\partial n$ are exactly the same as they would be in the absence of the screen. This condition allows us to specify the disturbance incident on the aperture by neglecting the presence of the screen.

(2) Over the portion of $S_1$ that lies in the geometrical shadow of the screen, the field distribution $U$ and its derivative $\partial U/\partial n$ are identically zero. This condition allows us to neglect all of the surface of integration except that portion lying directly within the aperture itself.

We now have

$$U(P_0) = \frac{1}{4\pi}\iint_{\Sigma}\left(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n}\right)\mathrm{d}s. \tag{8}$$

Suppose either $G$ or $\partial G/\partial n$ vanishes on $S_1$. Let the Green's function is written in terms of $P_0$ and its mirror image $\tilde{P}_0$ (where $\tilde{r}_{01}$ is the distance between $\tilde{P}_0$ and $P_1$), where the sources at these two points are $\pi$ rad out of phase:

$$G_-(P_1) = \frac{\exp(jkr_{01})}{r_{01}} - \frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}. \tag{9}$$

In this case, $G_-$ vanishes on $S_1$. We then have the first Rayleigh–Sommerfeld solution:

$$U(P_0) = -\frac{1}{4\pi}\iint_{\Sigma} U\frac{\partial G_-}{\partial n}\,\mathrm{d}s. \tag{10}$$

[1]While the Kirchhoff boundary conditions simplify the results considerably, it is important to realize that neither can be exactly true. The presence of the screen will inevitably perturb the fields on $\Sigma$ to some degree, for along the rim of the aperture certain boundary conditions must be met that would not be required in the absence of the screen. In addition, the shadow behind the screen is never perfect, for fields will inevitably extend behind the screen for a distance of several wavelengths. However, if the dimensions of the aperture are large compared with a wavelength, these fringing effects can be safely neglected, and the two boundary conditions can be used to yield results that agree very well with experiment.

The normal derivative of $G_-$ is

$$\frac{\partial G_-(P_1)}{\partial n} = \cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01})\left(jk - \frac{1}{r_{01}}\right)\frac{\exp(jkr_{01})}{r_{01}} - \cos(\,\hat{\mathbf{n}}, \tilde{\mathbf{r}}_{01})\left(jk - \frac{1}{\tilde{r}_{01}}\right)\frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}. \quad (11)$$

For $P_1$ on $S_1$, $\mathbf{r}_{01} = \tilde{\mathbf{r}}_{01}$ and $\cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01}) = -\cos(\,\hat{\mathbf{n}}, \tilde{\mathbf{r}}_{01})$, so on $S_1$,

$$\frac{\partial G_-(P_1)}{\partial n} = 2\cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01})\left(jk - \frac{1}{r_{01}}\right)\frac{\exp(jkr_{01})}{r_{01}}. \quad (12)$$

When $r_{01} \gg \lambda$,

$$\frac{\partial G_-(P_1)}{\partial n} = 2jk\cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01})\frac{\exp(jkr_{01})}{r_{01}}. \quad (13)$$

Substituting this into the first Rayleigh–Sommerfeld solution, we have the Huygens–Fresnel principle:

$$U(P_0) = \frac{1}{j\lambda}\iint_\Sigma U(P_1)\frac{\exp(jkr_{01})}{r_{01}}\cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01})\,\mathrm{d}s. \quad (14)$$

This integral expresses the observed field $U(P_0)$ as a superposition of diverging spherical waves $\exp(jkr_{01})/r_{01}$ originating from secondary sources located at each and every point $P_1$ within the aperture.

We can express the obliquity factor exactly as $\cos(\,\hat{\mathbf{n}}, \mathbf{r}_{01}) = \Delta z/r_{01}$, and represent $P_0$ with the transverse coordinates $\mathbf{u} = (u, v)$ and $P_1$ with the transverse coordinates $\mathbf{u}_0 = (u_0, v_0)$. Then

$$U(\mathbf{u}, z) = \frac{\Delta z}{j\lambda}\iint_\Sigma U(\mathbf{u}_0, z_0)\frac{\exp(jkr_{01})}{r_{01}^2}\,\mathrm{d}\mathbf{u}_0, \quad (15)$$

where $r_{01} = \sqrt{(\Delta z)^2 + \|\Delta\mathbf{u}\|^2}$.

## 2. The Fresnel approximation

The Fresnel approximation is made via application of the binomial expansion ($\sqrt{1+b} \approx 1 + b/2 - b^2/8 + \cdots$):

$$r_{01} = \Delta z\sqrt{1 + \frac{\|\Delta\mathbf{u}\|^2}{(\Delta z)^2}} \approx \Delta z\left(1 + \frac{\|\Delta\mathbf{u}\|^2}{2(\Delta z)^2}\right) = \Delta z + \frac{\|\Delta\mathbf{u}\|^2}{2\Delta z}. \quad (16)$$

Only the first term is required for the approximation of the $r_{01}^2$ that appears in the denominator, but both terms are required for the approximation of $r_{01}$ that appears in the exponent. This is because errors in the approximation of $r_{01}$ in the exponent are multiplied by a very large number $k$ ($k \sim 10^7$ for optical wavelengths), and small phase changes are very significant. Hence

$$U(\mathbf{u}, z) = \frac{e^{jk\Delta z}}{j\lambda\Delta z}\iint_\Sigma U_0(\mathbf{u_0}, z_0)\exp\left(jk\frac{\|\Delta\mathbf{u}\|^2}{2\Delta z}\right)\mathrm{d}\mathbf{u_0}. \quad (17)$$

## 3. Two-dimensional fractional Fourier transform

The two-dimensional fractional Fourier transform is

$$\mathcal{F}_\alpha\{f\}(\mathbf{x}) = \frac{e^{j(\pi\hat{\alpha}-2\alpha)/2}}{2\pi|\sin\alpha|}\exp\left(-\frac{j}{2}\cot\alpha\|\mathbf{x}\|^2\right)\iint\exp\left(j\frac{\mathbf{x}\cdot\mathbf{x}'}{\sin\alpha}-\frac{j}{2}\cot\alpha\|\mathbf{x}'\|^2\right)f(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{18}$$

where the integrals are infinite, $0<|\alpha|<\pi$, and $\hat{\alpha}=\operatorname{sgn}(\sin\alpha)$ [3, 2].

## 4. Fresnel diffraction as a fractional Fourier transform

To show that the Fresnel diffraction integral (Eq. 17) involves a fractional Fourier transform, some scaling is necessary. Let $\ell$ be a characteristic length such that $\|\mathbf{u}_0\|=\ell\|\mathbf{x}'\|$. Equating the corresponding exponents in Eqs. 17 and 18,

$$\frac{k}{\Delta z}\|\mathbf{u}_0\|^2 = \cot\alpha\|\mathbf{x}'\|^2 \tag{19}$$

$$\|\mathbf{u}_0\|^2 = \frac{\Delta z}{k}\cot\alpha\|\mathbf{x}'\|^2 \tag{20}$$

$$\ell = \sqrt{(\Delta z/k)\cot\alpha}. \tag{21}$$

Similarly,

$$\frac{k}{\Delta z}\mathbf{u}\cdot\mathbf{u}_0 = \frac{\mathbf{x}\cdot\mathbf{x}'}{\sin\alpha} \tag{22}$$

$$= \frac{\mathbf{x}\cdot\mathbf{u}_0}{\sin\alpha}\frac{\|\mathbf{x}'\|}{\|\mathbf{u}_0\|} \tag{23}$$

$$\frac{k}{\Delta z}\|\mathbf{u}\| = \frac{\|\mathbf{x}\|}{\sin\alpha}\frac{1}{\ell} \tag{24}$$

$$\|\mathbf{u}\| = \frac{\Delta z}{k}\sqrt{\frac{k\tan\alpha}{\Delta z\sin^2\alpha}}\|\mathbf{x}\| \tag{25}$$

$$= \sqrt{(\Delta z/k)\cot\alpha}\cos\alpha\|\mathbf{x}\| \tag{26}$$

$$\|\mathbf{u}\| = \ell\cos\alpha\|\mathbf{x}\|. \tag{27}$$

Note that $\mathrm{d}\mathbf{u}_0=\mathrm{d}u_0\,\mathrm{d}v_0$, so $\mathrm{d}\mathbf{u}_0=(\Delta z/k)\cot\alpha\,\mathrm{d}\mathbf{x}'$. Let $\tilde{U}_0(\mathbf{x}',z)=U_0(\ell\mathbf{x}')$. Then

$$U(\mathbf{u},z) = \frac{e^{jk\Delta z}}{j2\pi}\cot\alpha\exp\left(\frac{j}{2}\frac{\cot\alpha}{\cos^2\alpha}\|\mathbf{x}\|^2\right)\iint_\Sigma\tilde{U}_0(\mathbf{x}',z)\exp\left(-j\frac{\mathbf{x}\cdot\mathbf{x}'}{\sin\alpha}+\frac{j}{2}\cot\alpha\|\mathbf{x}'\|^2\right)\mathrm{d}\mathbf{x}'. \tag{28}$$

Given the trigonometric identity

$$\frac{\cot\alpha}{\cos^2\alpha} = \frac{1}{\sin\alpha\cos\alpha} = \frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha} = \tan\alpha+\cot\alpha,$$

$$U(\mathbf{u},z) = \frac{e^{jk\Delta z}}{j2\pi}\cot\alpha\exp\left(\frac{j}{2}(\tan\alpha+\cot\alpha)\|\mathbf{x}\|^2\right) \times\iint_\Sigma\tilde{U}_0(\mathbf{x}',z)\exp\left(-j\frac{\mathbf{x}\cdot\mathbf{x}'}{\sin\alpha}+\frac{j}{2}\cot\alpha\|\mathbf{x}'\|^2\right)\mathrm{d}\mathbf{x}'. \tag{29}$$

Using the complex conjugate of Eq. 18, $\mathcal{F}_{-\alpha}\{f\}(\mathbf{x})$, then

$$\frac{2\pi|\sin\alpha|}{e^{-j(\pi\hat{\alpha}-2\alpha)/2}}\mathcal{F}_{-\alpha}\{f\}(\mathbf{x}) = \exp\left(\frac{j}{2}\cot\alpha\|\mathbf{x}\|^2\right)\iint \exp\left(-j\frac{\mathbf{x}\cdot\mathbf{x}'}{\sin\alpha} + \frac{j}{2}\cot\alpha\|\mathbf{x}'\|^2\right) f(\mathbf{x}')\,\mathrm{d}\mathbf{x}'. \tag{30}$$

Substituting this result into Eq. 29,

$$U(\mathbf{u}, z) = e^{jk\Delta z}\frac{\cot\alpha|\sin\alpha|}{je^{-j(\pi\hat{\alpha}-2\alpha)/2}}\exp\left(\frac{j}{2}\tan\alpha\|\mathbf{x}\|^2\right)\mathcal{F}_{-\alpha}\{\tilde{U}_0\}(\mathbf{x}, z) \tag{31}$$

$$= \frac{e^{j(k\Delta z-\alpha)}}{je^{-j\pi\hat{\alpha}/2}}\hat{\alpha}\cos\alpha\exp\left(\frac{jk}{2\Delta z}\sin^2\alpha\|\mathbf{u}\|^2\right)\mathcal{F}_{-\alpha}\{\tilde{U}_0\}(\mathbf{x}, z), \tag{32}$$

where the fact that $\hat{\alpha} = |\sin\alpha|/\sin\alpha$ and Eq. 27 have both been used in the second line.

When $\alpha > 0$, $\hat{\alpha} = 1$ and $je^{-j\pi/2} = 1$. Under this condition,

$$U(\mathbf{u}, z) = e^{j(k\Delta z-\alpha)}\cos\alpha\exp\left(\frac{jk}{2\Delta z}\sin^2\alpha\|\mathbf{u}\|^2\right)\mathcal{F}_{-\alpha}\{\tilde{U}_0\}(\mathbf{x}, z). \tag{33}$$

Hence Fresnel propagation over a distance $\Delta z$ involves a fractional Fourier transform, scaled by a few factors. The $e^{jk\Delta z}$ represents the constant-phase contribution that is often neglected, the $e^{j\alpha}$ term is the Gouy phase shift (see §8.8.4, p. 498 of [8], or p. 86 of [9]), and the remaining exponential represents a quadratic approximation to a diverging spherical wavefront with radius $\mathscr{R}_\alpha = \Delta z/\sin^2\alpha$.

## References


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